1 General vector spaces

Definition 1. Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by numbers called scalars. By addition we mean a rule for associating with each pair of objects u and v in V an object $u + v$, called the sum of u and v; by scalar multiplication we mean a rule for associating with each scalar k and each object $u \in V$ an object ku, called the scalar multiple of u by k. If the following axioms are satisfied by all objects $u, v, w \in V$ and all scalars k, m, then we call V a vector space and we call the objects in V vectors.

(1) If u and v are objects in$V$, then $u + v \in V$. (closed under vector addition)

(2) $u + v = v + u$.

(3) $u + (v + w) = (u + v) + w$.

(4) There is an object $\vec{0} \in V$, called a zero vector for V, such that $u + \vec{0} = \vec{0} + u = u$ for all $u \in V$.

(5) For each u in V, there is an object $-u$ in V, called a negative of u, such that $u + (-u) = (-u) + u = \vec{0}$.

(6) If k is any scalar and u is any object in V, then ku is in V. (closed under scalar multiplication)

(7) $k(u + v) = ku + kv$.

(8) $(k + m)u = ku + mu$.

(9) $k(mu) = (km)(u)$.

(10) $1u = u$.

Example 2. The set $V = \mathbb{R}^n$ can be naturally made into a vector space with the usual operations:

$$u + v = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n),$$

$$kv = k(x_1, \ldots, x_n) = (kx_1, \ldots, kx_n).$$

We can check that axioms (1) to (10) are satisfied.

Example 3. Consider the set V of all $m \times n$ matrices with the usual matrix operations of addition and scalar multiplication. These operations satisfy the actions of vector spaces and we have another example of vector spaces, denoted in this case $M_{m\times n}$. On the other hand if we restrict ourselves to invertible matrix, the zero matrix $\vec{0} = (0)$ is not containing in the set and we do not have a vector space.

Example 4. Let V be the set of real-valued functions that are defined at each $x \in \mathbb{R} = (-\infty, \infty)$. If $f = f(x)$ and $g = g(x)$ are two functions in V and if k is any scalar, then define the operations of addition and scalar multiplication by

$$(f+g)(x) = f(x) + g(x) \qquad\qquad (kf)(x) = kf(x).$$

The vector space defined this way is denoted $F(-\infty, \infty)$.

Example 5. The set S of all pairs of real numbers of the form $(x, y) \in \mathbb{R}^2$, where $x \geq 0$, with the standard operations on each component is not a vector space. The reason is failure of axiom (5). For example, the element $v = (3, -4) \in S$, but the $-v = (-3, -4) \notin S$, since $x < 0$.

Example 6. The sphere of radius R in $\mathbb{R}^3$ is given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq R^2\}.$$

It is not a vector space for the usual operations in $\mathbb{R}^3$ for several reasons. It is not closed under vector's addition and is also not closed under scalar multiplication.

Example 7. The subset of $\mathbb{R}^3$ given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x = y = z\}.$$

It is a vector space for the usual operations in $\mathbb{R}^3$. You can check that it satisfies the axioms.

Example 8. In general the solution set of a system of homogeneous linear equations in n variables,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \qquad\qquad\qquad\qquad\vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0 \\ \qquad\qquad\qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

is a vector space inside the vector space $\mathbb{R}^n$. For example:

$$\begin{cases} 4w + 3x + y - z = 0 \\ -3w + x - 2y + z = 0 \end{cases}$$

is a real vector space. In this case, a vector space in $\mathbb{R}^4$.

Proposition 9. *Let V a vector space and $v \in V$ a vector. Then we have* *(a)* $0v = \vec{0}$.

(b) $(-1)v = -v$.

Proof. For part (i), using (8) $(k+m)u = ku+mu \Rightarrow (0+0)u = 0u+0u \Rightarrow 0u = 0u+0u$. Adding $-0u$ to both sides of the equation, we get $\vec{0} = 0u + \vec{0} = 0u$.
(ii) $(-1)v + v = (-1)v + 1v = (-1+1)v = 0v = \vec{0}$. Hence $(-1)v = -v$. $\square$

Definition 10. A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Example 11. As observed before, the solution set W of a system of homogeneous linear equations in n variables,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

is a vector space inside the vector space $V = \mathbb{R}^n$. We can observe that: The sum of two vectors in W is still in W and the multiplication of a vector in W by a scalar k is still a vector $kw \in W$. The fact that the system is homogeneous is crucial for this second property, since the solution to any system of equations (linear variety) will be closed under addition.

Proposition 12. *If W is a set of one or more vectors in a vector space V,then W is a subspace of V if and only if the following conditions are satisfied.*
(a) If u, v are vectors in W, then $u + v \in W$.
(b) If k is a scalar and v is a vector in W, then kv is also a vector in W.

Proof. The axioms of vector space for W gives in particular properties (a) and (b). On the other hand if we have the axioms of vector space for V and properties (a) and (b), the only thing left to prove is that:
(4) The zero vector is in W: By (b) $0w = \vec{0} \in W$, for any $w \in W$.
(5) The opposite of a vector in W is also in W: Also by (b) $(-1)w = -w \in W$ for any $w \in W$. $\square$

Example 13. Differentiable functions on (a, b) are a subspace of the vector space $F(a, b)$ of functions $f\colon (a, b) \longrightarrow \mathbb{R}$.

Example 14. Symmetric matrices are a subspace in the vector space of square matrices $M_n(\mathbb{R})$.

Example 15. Polynomial functions is a subspace of the vector space of $F(-\infty,\infty)$. In particular, we have the subspace P_n of polynomials of degree at most n.

Definition 16. If w is a vector in a vector space V, then w is said to be a linear combination of the vectors $v_1, v_2, \dots, v_r \in V$ if w can be expressed in the form

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r,$$

where $k_1, k_2, \dots, k_3$ are scalars. These scalars are called the coefficients of the linear combination.

Theorem 17. *If $S = \{w_1, w_2, ..., w_r\}$ is a nonempty set of vectors in a vector space V, then:*

(a) The set W of all possible linear combinations of the vectors in S is a subspace of V

(b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains also W.

Proof. We should prove that W is closed under addition and scalar multiplication. For example for addition, take two elements $w, w' \in W$, then the sum

$$\begin{aligned} w + w' &= k_1 v_1 + k_2 v_2 + \cdots + k_r v_r + k'_1 v_1 + k'_2 v_2 + \cdots + k'_r v_r \\ &= (k_1 + k'_1) v_1 + (k_2 + k'_2) v_2 + \cdots + (k_r + k'_r) v_r, \end{aligned}$$

will also be an element of W. In a similar way, we can do closed by scalar multiplication. □

Definition 18. If $S = \{w_1, w_2, \dots, w_r\}$ is a nonempty set of vectors in a vector space V, then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V generated by S, and we say that the vectors $w_1, w_2, \dots, w_r$ span W. We denote this subspace as $W = \operatorname{span}\{w_1, w_2, \dots, w_r\}$ or $W = \operatorname{span}(S)$.

Example 19. The vectors subspace P_n of polynomials of degree at most n is exactly the vector subspace $P_n = \operatorname{span}\{1, x, \dots, x^n\}$ spanned or generated by the polynomials $1, x, x^2, \dots, x^n$.

Remark 20. Suppose that $w_1 = (a_{1,1}, \dots a_{1,n}), \dots, w_r = (a_{1,r}, \dots a_{n,r})$ are vectors in $\mathbb{R}^n$. In order to determine whether or not a vector $b = (b_1, \dots, b_r)$ is in

$\text{span}\{w_1, w_2, \ldots, w_r\}$, we need to solve the linear system of equations:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n = b_i \\ \vdots \\ a_{r,1}x_1 + a_{r,2}x_2 + \cdots + a_{r,n}x_n = b_r. \end{cases}$$

Definition 21. If $S = \{v_1, v_2, \ldots v_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a linearly independent set if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be linearly dependent.

Theorem 22. *A nonempty set $S = \{v_1, v_2, \ldots v_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation*

$$k_1v_1 + k_2v_2 + \cdots + k_rv_r = 0$$

are $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.

Proof. If the vectors in $S = \{v_1, v_2, \ldots v_r\}$ are linearly dependent, then there exist a $v_i = k_1v_1 + \cdots + k_vv_r$. The linear combination $-v_i + k_1v_1 + \cdots + k_vv_r = 0$ and not all coefficients are zero. On the other hand if $k_1v_1 + k_2v_2 + \cdots + k_rv_r = 0$ and there is $k_i \neq 0$, then $v_i = k_1/k_iv_1 + \ldots k_r/k_iv_r$ is a linear combination of the other vectors and the vectors in S are linearly dependent. □

Remark 23. Suppose that $w_1 = (a_{1,1}, \ldots a_{1,n}), \ldots, w_r = (a_{1,r}, \ldots a_{n,r})$ are vectors in $\mathbb{R}^n$. The vectors in the set $S = \{w_1, w_2, \ldots, w_r\}$ are linearly independent if and only if the homogeneous system of equations:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = 0 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n = 0 \\ \vdots \\ a_{r,1}x_1 + a_{r,2}x_2 + \cdots + a_{r,n}x_n = 0, \end{cases}$$

has a unique solution $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0) \in \mathbb{R}^n$. In the special case of $r = n$, this is equivalent to show that the determinant $\det(A)$ of the matrix A of the system satisfies $\det(A) \neq 0$.

Theorem 24. *Let $S = \{v_1, v_2, \ldots v_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$ then vectors in S are linearly dependent.*

Proof. To check whether or not the vectors are linearly dependent we end up with a homogeneous system of n equations in the r unknowns $k_1, \ldots, k_r$. Since $r > n$, it follows that the system has nontrivial solutions. $\square$

Definition 25. If $S = \{v_1, v_2, \ldots, v_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a basis for V if:

(a) S span V.

(b) S is linearly independent.

The dimension of V is the number of vectors in a basis for V.

Theorem 26. *(Uniqueness of Basis Representation) If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then every vector v in V can be expressed in the form*

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

in exactly one way. The scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of v relative to the basis S.

Theorem 27. *Let V be an n-dimensional vector space, and let $\{v_1, v_2, \ldots, v_n\}$ be any basis. Then:*
(a) If a set in V has more than n vectors, then it is linearly dependent.
(b) If a set in V has fewer than n vectors, then it does not span V.

If we change the basis for a vector space V from an old basis $B = \{u_1, u_2, \ldots, u_n\}$ to a new basis $B' = \{u'_1, u'_2, \ldots, u'_n\}$, then for each vector $v \in V$, the old coordinate vector $[v]_B$ is related to the new coordinate vector $[v]_{B'}$ by the equation

$$[v]_B = P[v]_{B'},$$

where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is, the column vectors of P are $\{[u'_1]_B, [u'_2]_B, ..., [u'_n]_B\}$.

Definition 28. If A is an $m \times n$ matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the **row space** of A, and the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the **column space** of A. The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of $\mathbb{R}^n$, is called the **null space** of A.

Theorem 29. *If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's form a basis for the column space of R.*

Definition 30. The column space and the row space have the same dimension. The common dimension of the row space and column space of a matrix A is called the **rank of A** and is denoted by rank(A); the dimension of the null space of A is called the **nullity of A** and is denoted by nullity(A).